Review

- Transformations is when we change the basic graph of a function in 2-dimensional space
- In this section, we will look at:
  - **Translations** – vertical and horizontal shifts
  - **Compression and Expansion** – stretch and squeeze
  - **Reflections** – in both the x *and* y *axes*
- If we consider a basic function: $y = f(x)$

This can seem a little daunting, so we will look at it piecewise.

Transformations can give us shifts represented by: $y = af[b(x - c)] + d$

1. **Translations**, or shifts, are additions or subtractions **represented by c and d**
2. **Expansions**, or compressions, are multiplications **shown by a and b**
3. **Reflections** happen **when a or b are negative**

- **Constants a and d**, which are **"outside of the function"**, affect the $y - values$ of the ordered pairs
- **Constants b and c**, which are **"inside the function"**, affect the $x - values$ of the ordered pairs

This is a big deal and can help us make this process as simple as possible!!

- Let's look at these various transformations separately.

Translations

A translation is when the graph **is shifted in the left or right (x direction)** or the **up and down (y direction)**, without changing the shape of the original graph

a) Vertical Translations (y direction), $d > 0$

If $d > 0$, for the graph of $y = f(x)$, the graph of:

$y = f(x) + d$ **is shifted up "d" units**

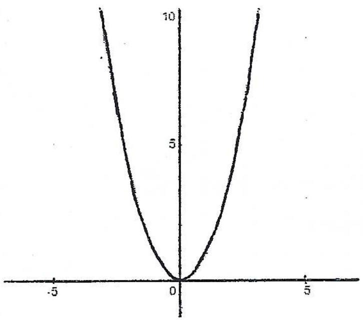
$y = f(x) - d$ **is shifted down "d" units**

Vertical Translations are quite **intuitive**, they literally **move up or down** depending of the **sign and number of the d value**

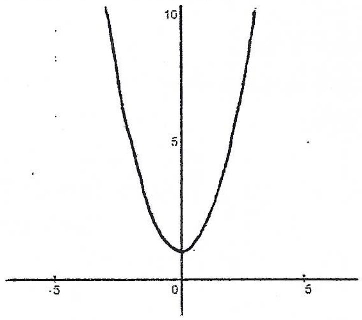
See the following graphs as examples of vertical translations

Example 1:

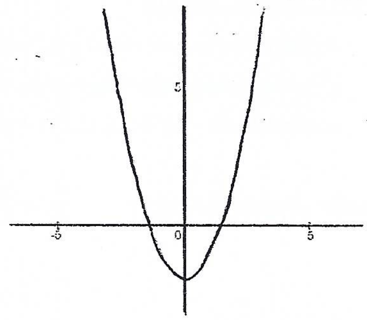
Quadratic Graphs



$y = x^2$
$y = f(x)$

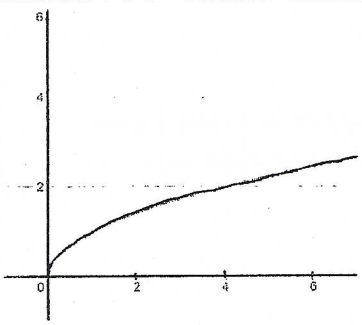


$y = x^2 + 1$
$y = f(x) + 1$

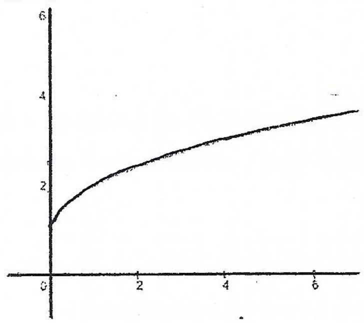


$y = x^2 - 2$
$y = f(x) - 2$

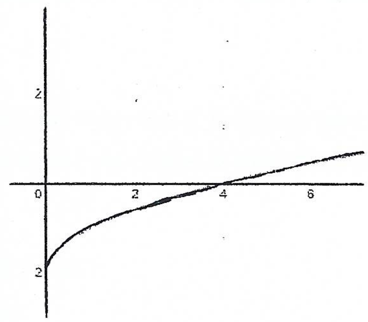
Square Root Graphs



$y = \sqrt{x}$
$y = f(x)$

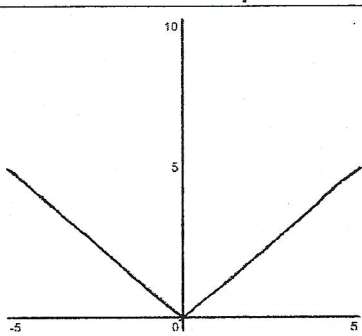


$y = \sqrt{x} + 1$
$y = f(x) + 1$

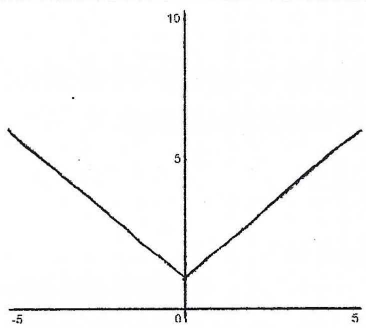


$y = \sqrt{x} - 2$
$y = f(x) - 2$

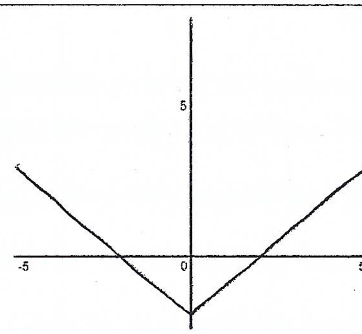
Absolute Value Graphs



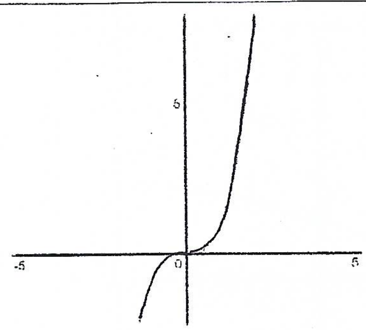
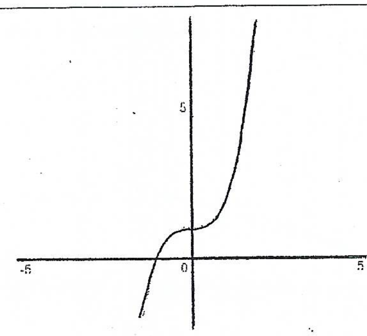
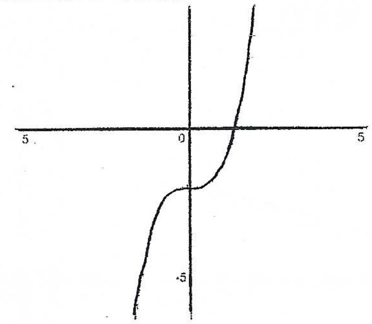
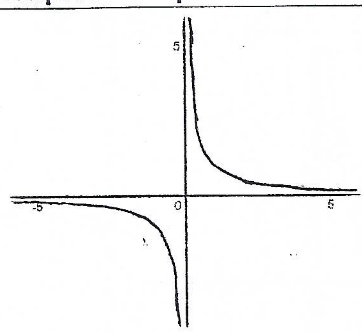
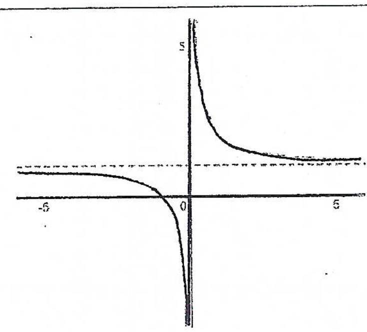
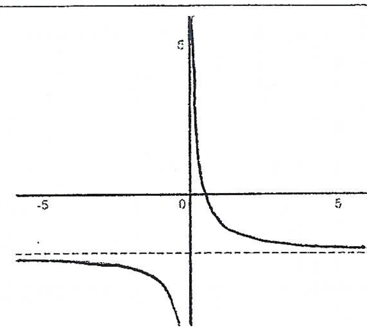
$y = |x|$
$y = f(x)$



$y = |x| + 1$
$y = f(x) + 1$



$y = |x| - 2$
$y = f(x) - 2$

Cubic Graphs		
 $y = x^3$ $y = f(x)$	 $y = x^3 + 1$ $y = f(x) + 1$	 $y = x^3 - 2$ $y = f(x) - 2$
Reciprocal Graphs		
 $y = \frac{1}{x}$ $y = f(x)$	 $y = \frac{1}{x} + 1$ $y = f(x)$	 $y = \frac{1}{x} - 2$ $y = f(x)$

b) Horizontal Translations (x $direction$), $c > 0$

If $c > 0$, for the graph of $y = f(x)$, the graph of:

$y = f(x + c)$ is **shifted left "c" units**

$y = f(x - c)$ is **shifted right "c" units**

Horizontal Translations are not **intuitive**, they **move the opposite direction** of the **sign of the c $value$**

I like to think to consider "what value of x makes the inside zero". That value is where you move on the $x - axis$.

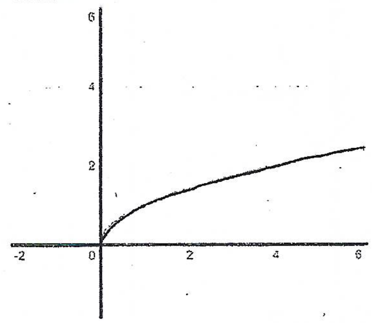
$y = f(x - 3)$ $\quad or \quad$ $y = f(x + 2)$

Moves right 3, or $x = 3$ makes $x - 3 = 0$

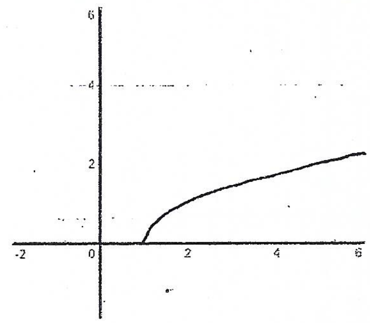
Moves left 2, or $x = -2$ makes $x + 2 = 0$

Example 2:

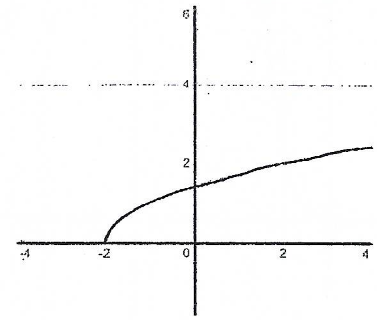
Square Root Graphs



$y = \sqrt{x}$
$y = f(x)$

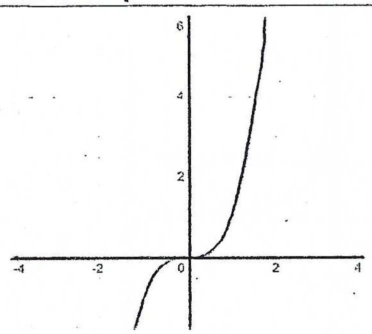


$y = \sqrt{x-1}$
$y = f(x-1)$

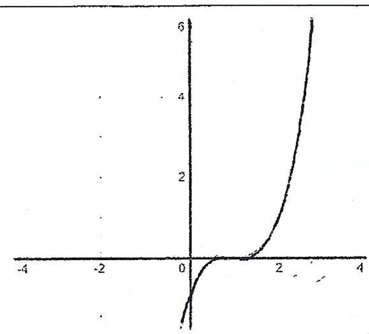


$y = \sqrt{x+2}$
$y = f(x+2)$

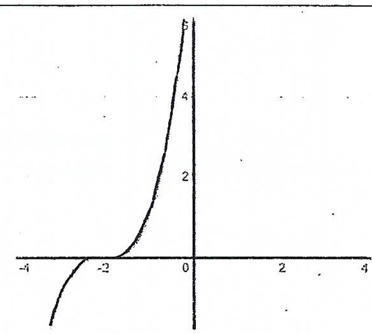
Cubic Graphs



$y = x^3$
$y = f(x)$



$y = (x-1)^3$
$y = f(x-1)$



$y = (x+2)^3$
$y = f(x+2)$

Summary

Vertical and Horizontal Translations of $y = f(x)$ with point (x, y) If $c, d > 0$:	
1. Vertical translation of d units *upward*	$h(x) = f(x) + d,\ \ (x, \mathbf{y + d})$
2. Vertical translation of d units *downward*	$h(x) = f(x) - d,\ \ (x, \mathbf{y - d})$
3. Horizontal translation of c units *to the right*	$h(x) = f(x - c),\ \ (\mathbf{x + c}, y)$
4. Horizontal translation of c units *to the left*	$h(x) = f(x + c),\ \ (\mathbf{x - c}, y)$

Example 3: Write the equation of the function $f(x) = \sqrt{x}$ after a transformation $4\ units\ right\ and\ 3\ units\ down$

Solution 3: $g(x) = \sqrt{x-4} - 3$ ← Three units down

↑ To the right means $x - 4$

Example 4: What transformations have occurred to change $y = f(x)$ into $y = f(x-2) + 4$?

Solution 4: Horizontal translation: $2\ units\ right$ Vertical Translation: $4\ units\ up$

Example 5: If $(2, 2)$ is in $y = f(x)$, which point is on $y = f(x+3) - 2$ ← This moves the $y - coordinate$ **down 2 units**

↑ This moves the $x - coordinate$ **left 3 units**

Solution 5: $(x - 3, y - 2)$

$(2 - 3, 2 - 2) \rightarrow (-1, 0)$

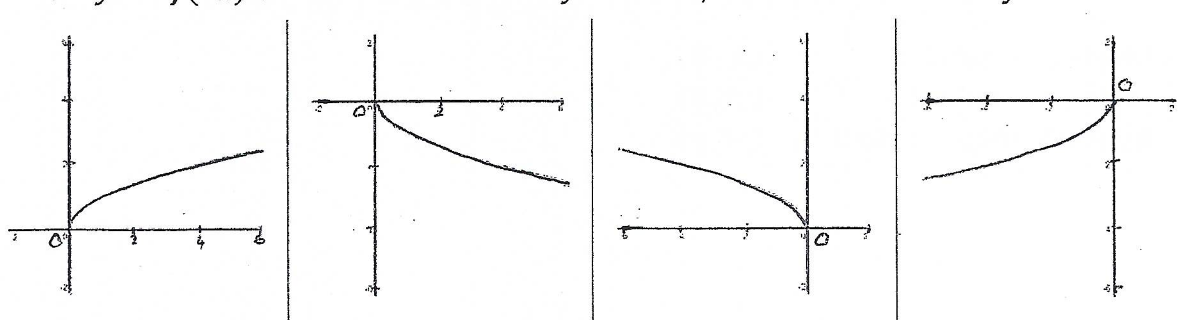
Reflections

The next type of transformation is a reflection. We are going to talk about reflecting over the $x - axis$ and $y - axis$ only.

- Consider reflecting over the $x - axis$, all $y - values$ change their signs.
- Consider reflecting over the $y - axis$, all $x - values$ change their signs.

For the graph of $y = f(x)$, the graph of:

- $y = -f(x)$ is a **refection of the** $y - values$, a **reflection in the** $x - axis$
- $y = f(-x)$ is a **refection of the** $x - values$, a **reflection in the** $y - axis$
- $y = -f(-x)$ is a **refection of the** $x\ and\ y - values$, a reflection in the $x\ and\ y - axis$



$y = \sqrt{x}$ $y = f(x)$ | $y = -\sqrt{x}$ $y = -f(x)$ | $y = \sqrt{-x}$ $y = f(-x)$ | $y = -\sqrt{-x}$ $y = -f(-x)$

Summary

Reflections of $y = f(x)$ with point (x, y) in the two Axes	
1. Reflection in the $x - axis$	$h(x) = -f(x),\ (x, -y)$
2. Reflection in the $y - axis$	$h(x) = f(-x),\ (-x, y)$
3. Reflection in both $axes$	$h(x) = -f(-x),\ (-x, -y)$

Example 6: Write the equation of the function $f(x) = x^2 + x$ if it is reflected in the:

a) $x - axis$
b) $y - axis$

Solution 6:

a) $f(x) \rightarrow -f(x)$ so $x^2 + x \rightarrow -(x^2 + x) = -x^2 - x$

b) $f(x) \rightarrow f(-x)$ so $x^2 + x \rightarrow (-x)^2 + (-x) = x^2 - x$

Example 7: What transformations have occurred to change $y = x^2 + 2x$ into $y = -(x^2 + 2x)$?

Solution 7: Since the entire original function is inside the brackets, the negative on the outside. It is a **reflection of the $y - values$ ($the\ x - axis$).**

Example 8: If $(3, 2)$ is in $y = f(x)$, which point is on:

a) $y = -f(x)$
b) $y = f(-x)$
c) $y = -f(-x)$

Solution 8:

a) Sign change in $y - values$: $(3, -2)$
b) Sign change in $x - values$: $(-3, 2)$
c) Sign change in $x\ and\ y - values$: $(-3, -2)$

Compression and Expansion of Graphs

- Vertical and horizontal shifts leave the shape of the graph the same
- Compressions and Expansions graph a shape change, either a squeeze of a stretch
- There are helpful markers to determine whether or not it is a Vertical or Horizontal stretch

a) Vertical Compression and Expansion

For the graph of $y = f(x)$, the graph of:

$y = a \cdot f(x)$ is a **Vertical Expansion if $a > 1$ (Expansion by a factor of a)**

$y = a \cdot f(x)$ is a **Vertical Compression if $0 < a < 1$ (Compression by a factor of a, where a is a proper fraction)**

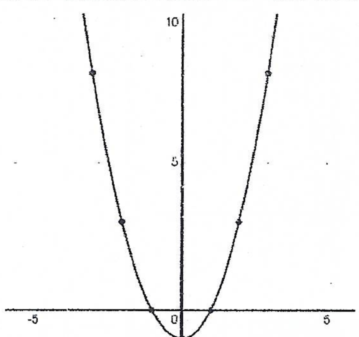
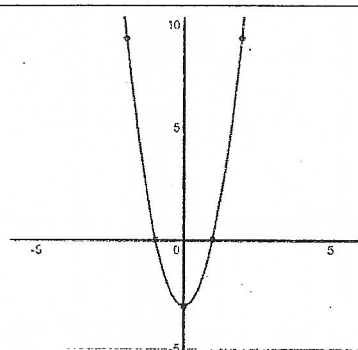
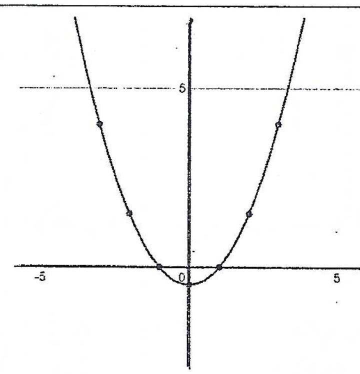
For the graph of $y = f(x)$, the graph of:

$y = 2f(x)$ is a **Vertical Expansion** by a **factor of 2**

$y = \frac{1}{3}f(x)$ is a **Vertical Compression** by a factor of $\frac{1}{3}$

Vertical Expansions and Compressions <u>keep the $x - intercepts$</u> of the original function!

<u>Example 11:</u>

Quadratic Graphs		
 $y = x^2 - 1$ $y = f(x)$ *Basic Graph*	 $y = 3(x^2 - 1)$ $y = 3f(x)$ *Vertical Expansion by a Factor of 3*	 $y = \frac{1}{2}(x^2 - 1)$ $y = \frac{1}{2}f(x)$ *Vertical Compression by a Factor of $\frac{1}{2}$*

You see the $x - intercepts$ did not change, but the shape of the graph was altered

b) **Horizontal Compressions and Expansion**

For the graph of $y = f(x)$, the graph of:

$y = f(bx)$ is a **Horizontal Compression** if $b > 1$ (by a factor of $\frac{1}{b}$)

$y = f(bx)$ is a **Horizontal Expansion** if $0 < b < 1$ (by a factor of $\frac{1}{b}$ where b is a proper fraction)

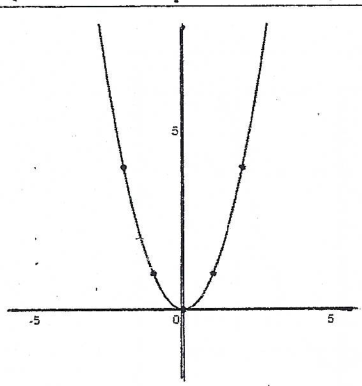
Horizontal Expansions and Compressions keep the $y-intercept$ of the original function!

For the graph of $y = f(x)$, the graph of:

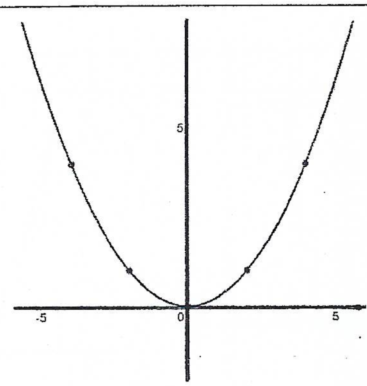
$y = f(2x)$ is a **Horizontal Compression** by a factor of $\frac{1}{2}$

$y = f(\frac{1}{3}x)$ is a **Horizontal Expansion** by a factor of 3

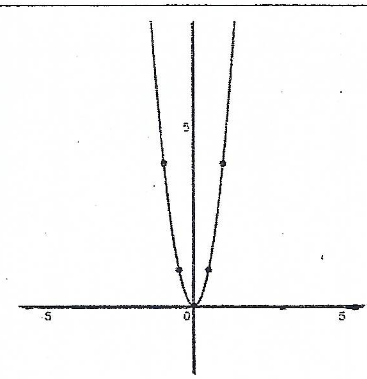
Example 11:

<table>
<tr><th colspan="3">Quadratic Graphs</th></tr>
<tr>
<td>

$y = x^2$
$y = f(x)$

Basic Graph</td>
<td>

$y = \left(\frac{1}{2}x\right)^2$
$y = f\left(\frac{1}{2}x\right)$

Horizontal Expansion by a Factor of 2</td>
<td>

$y = (2x)^2$
$y = f(2x)$

Horizontal Compression by a Factor of $\frac{1}{2}$</td>
</tr>
</table>

You see the $y - intercepts$ did not change, but the shape of the graph was altered

Summary

Vertical and Horizontal Compressions and Expansions of $y = f(x)$ with point (x, y) If $a > 1, b > 1$:	
1. Vertical expansion by a factor of a	$h(x) = af(x),\ (x, \boldsymbol{ay})$
2. Horizontal compressions by a factor of $\frac{1}{b}$	$h(x) = f(bx),\ (\frac{\mathbf{1}}{\boldsymbol{b}}x, y)$
If $0 < a < 1, 0 < b < 1$:	
3. Vertical expansion by a factor of a (a is a proper fraction)	$h(x) = af(x),\ (x, \boldsymbol{ay})$
4. Horizontal compressions by a factor of $\frac{1}{b}$ (b is the reciprocal of a proper fraction)	$h(x) = f(bx),\ (\boldsymbol{bx}, y)$

Example 12: Write an equation for the function $y = \sqrt{x}$, with a

a) Vertical Expansion by a factor of 2

b) Vertical Compression by a factor of $\frac{1}{2}$

c) Horizontal Expansion by a factor of 2

d) Horizontal Compression by a factor of $\frac{1}{2}$

Solution 12:

a) $y = 2\sqrt{x}$ b) $y = \frac{1}{2}\sqrt{x}$ c) $y = \sqrt{\frac{1}{2}x}$ d) $y = \sqrt{2x}$

Example 13: What transformation has happened to $y = f(x)$ to produce $y = 3f(\frac{1}{4}x)$?

Solution 13:

✓ **Vertical expansion** by a **factor of 3**

✓ **Horizontal expansion** by a **factor of** $\frac{\mathbf{1}}{\frac{1}{4}} \rightarrow \mathbf{4}$

Example 14: If $(3, 1)$ is on $y = f(x)$, what point is on $y = 2f(4x)$?

Solution 14:

$$(x, y) \rightarrow \left(\frac{1}{4}x, 2y\right) \rightarrow \left(\frac{1}{4}(3), 2(1)\right) \rightarrow \left(\frac{3}{4}, 2\right)$$

Practice Problems

1. Write an equation for the function that is described by the given characteristics.

a) The shape $f(x) = x^2$, moved 4 *units* to the left and 5 *units* downward.

b) The shape $f(x) = x^2$, moved 2 *units* to the right, reflected in the $x - axis$, and moved 3 *units* upward.

c) The shape $f(x) = x^3$, moved 2 *units* to the right and 3 *units* downward.

d) The shape $f(x) = x^3$, moved 1 *unit* downward and reflected in the $y - axis$.

e) The shape $f(x) = |x|$, moved 6 *units* upward and 3 *units* to the left.

f) The shape $f(x) = |x|$, moved 3 *units* to the left and reflected in the $x - axis$

g) The shape $f(x) = \sqrt{x}$, moved 7 *units* to the right and reflected in the $x - axis$

h) The shape $f(x) = \sqrt{x}$, moved 4 *units* upward and reflected in the $y - axis$

2. If $(-3,1)$ or (a,b) is a point on the graph of $y = f(x)$, what must be a point on the graph of the following?

a) $y = f(x+2)$

b) $y = f(x) + 2$

c) $y = f(x-2) - 2$

d) $y = -f(x)$

e) $y = f(-x)$

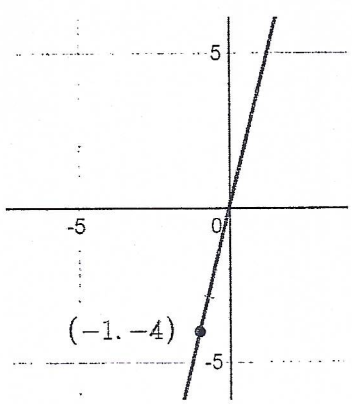
f) $y = -f(-x)$

g) $y = f(-x) - 2$

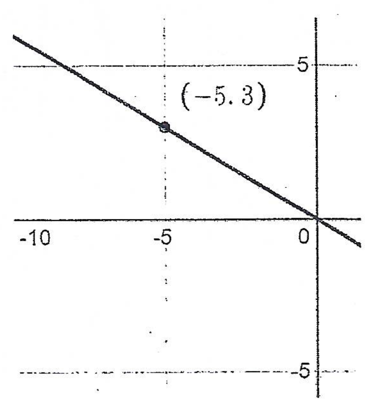
h) $y = -f(x+2)$

3. Use the graph of $f(x) = x$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

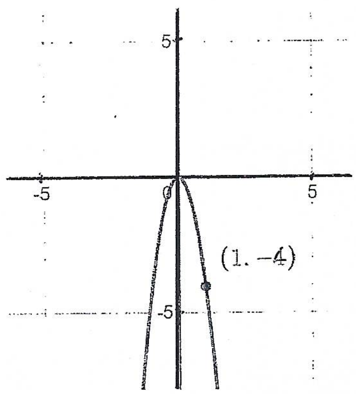


b)

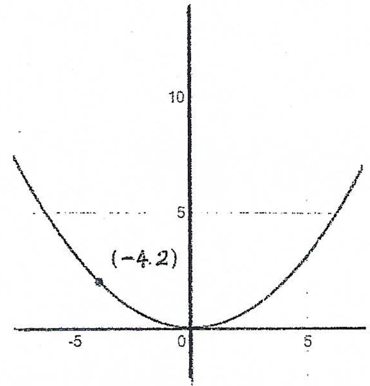


4. Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

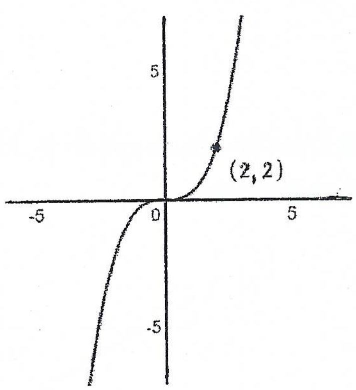


b)

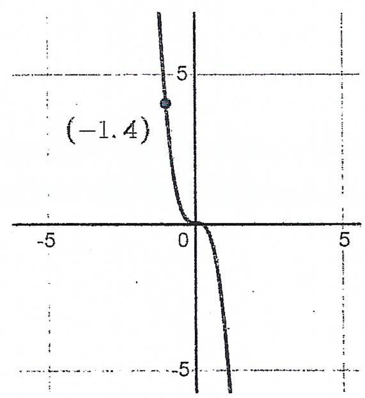


5. Use the graph of $f(x) = x^3$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

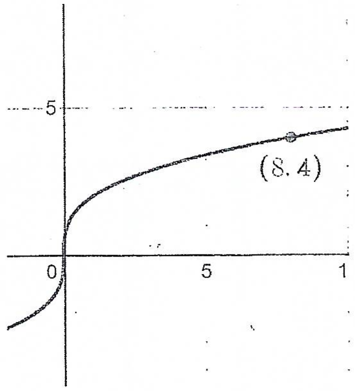


b)

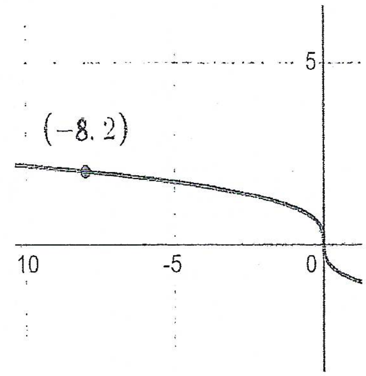


8. Use the graph of $f(x) = x^{\frac{1}{3}}$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

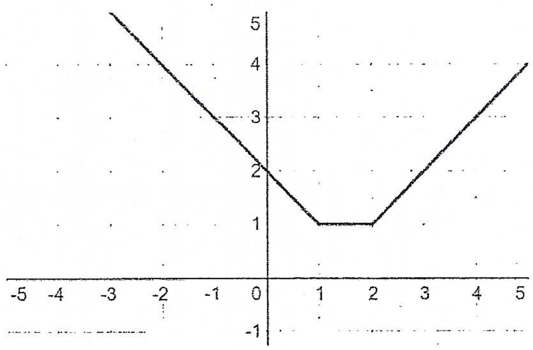
a)



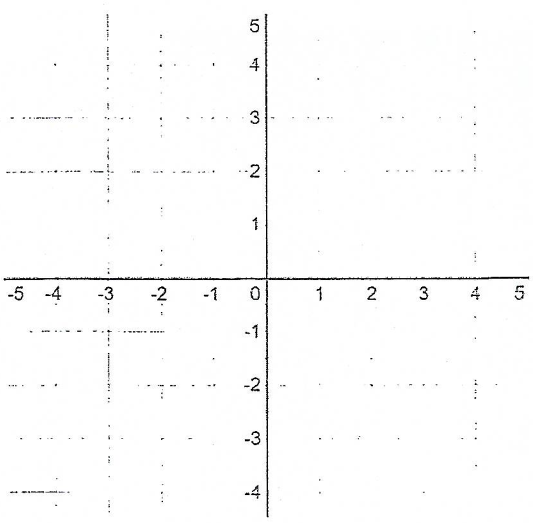
b)



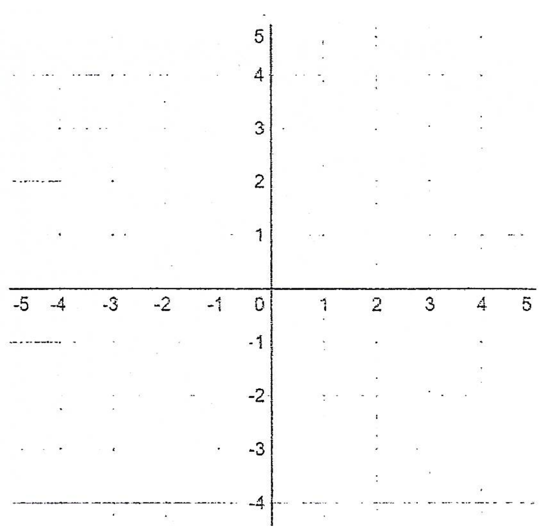
9. Given the graph of $f(x)$ below, sketch the graphs of the following:



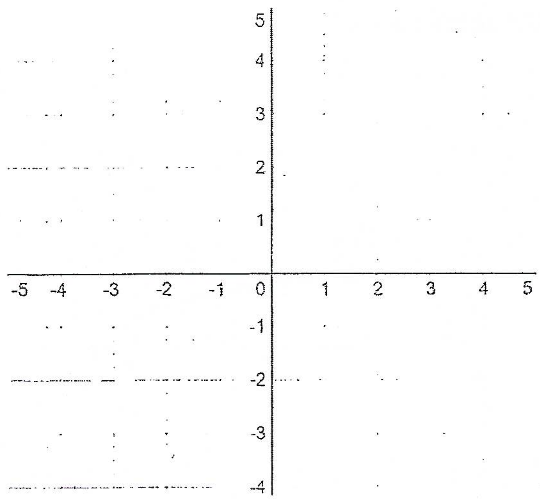
a) $y = -f(x)$



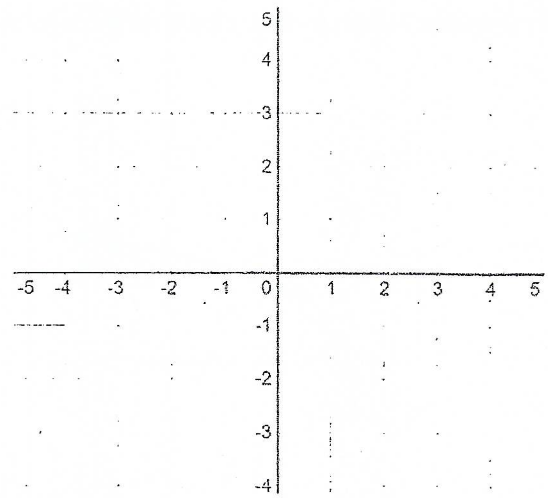
b) $y = f(-x)$



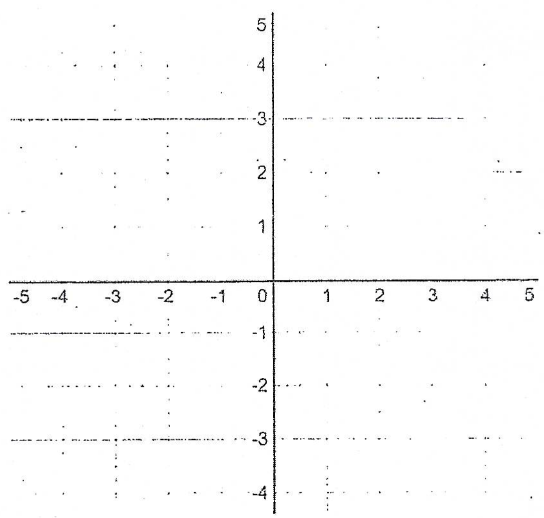
c) $y = -f(-x)$



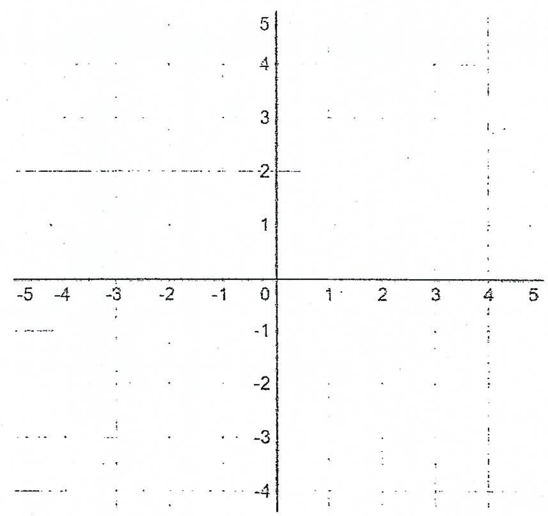
d) $y = f(x + 1)$



e) $y = f(x) - 2$



f) $y = f(1 - x)$



10. If $(-2, 4)$ is a point on the graph of $y = f(x - 1)$, what must be a point on the following graphs?

a) $y = f(x)$

b) $y = -f(x)$

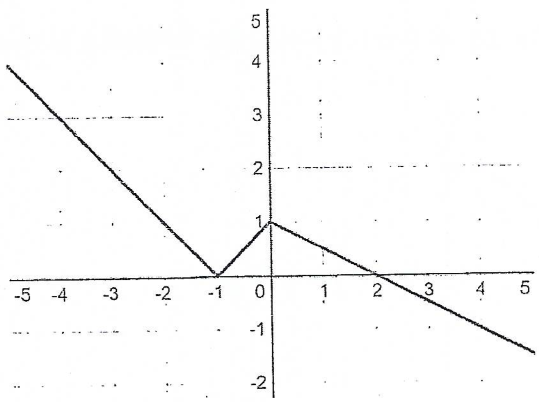
c) $y = f(-x)$

d) $y = f(x) + 2$

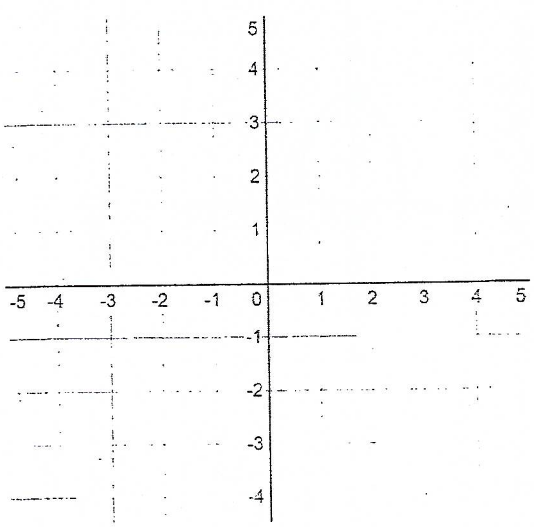
e) $y = f(x + 2)$

f) $y = -f(-x)$

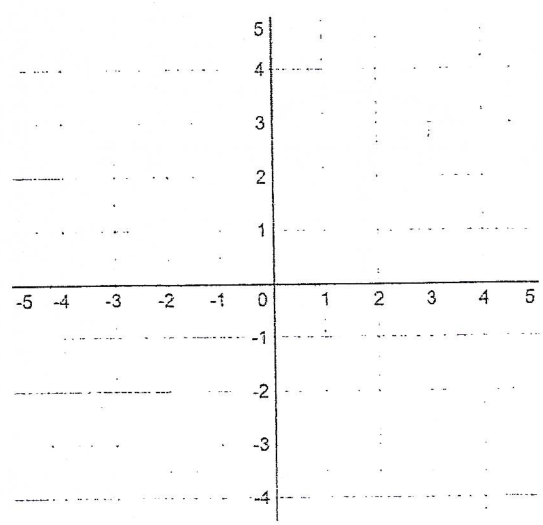
18. Given the graph of $f(x)$ below, sketch the graphs of the following:



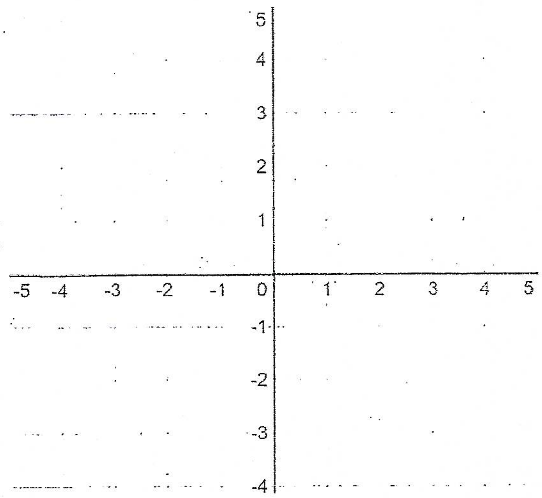
a) $y = 2f(x)$



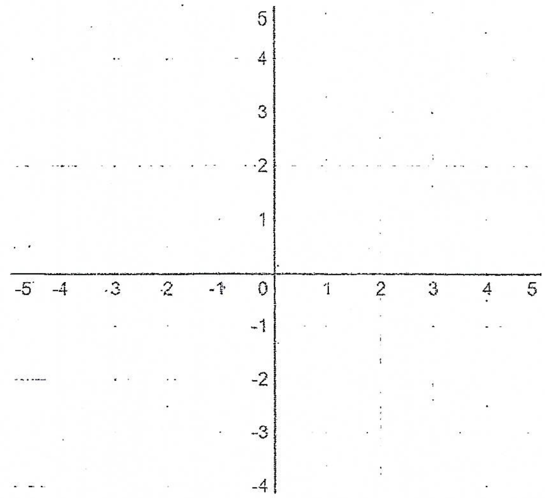
b) $y = f(2x)$



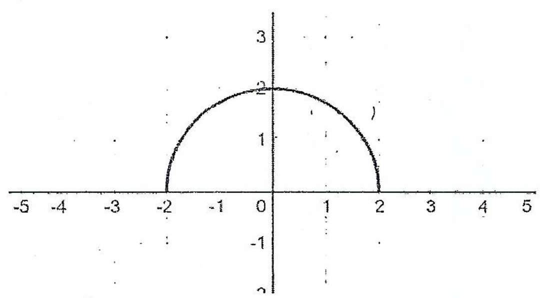
c) $y = -f\left(\frac{x}{2}\right)$



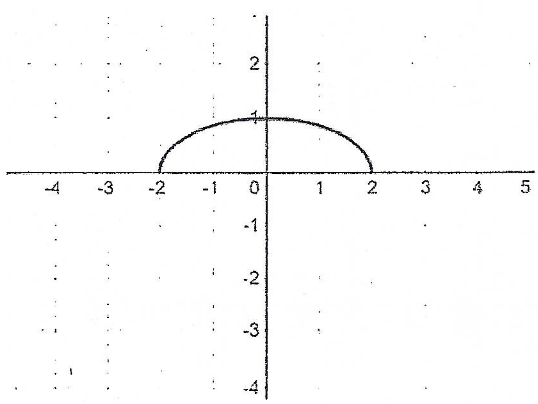
d) $y = -\frac{1}{2}f(-x)$



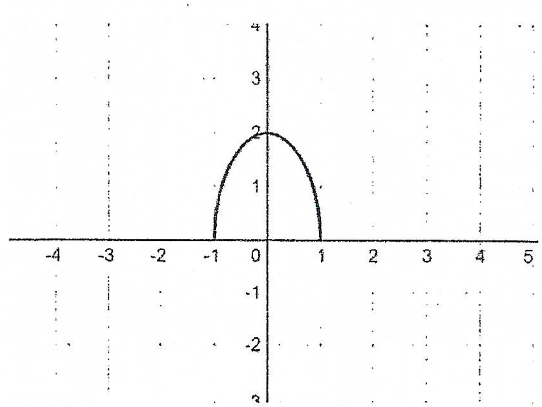
19. Given the graph of $f(x)$ below, what equations represent the following graphs



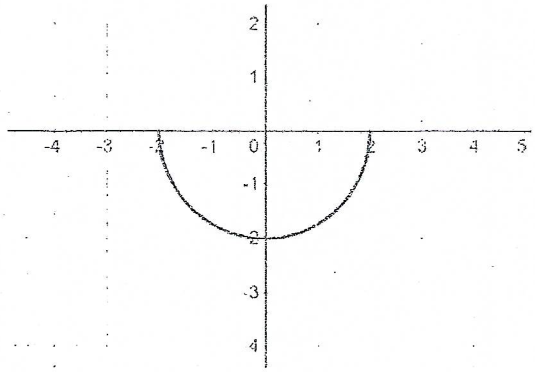
a) $y =$ ___________



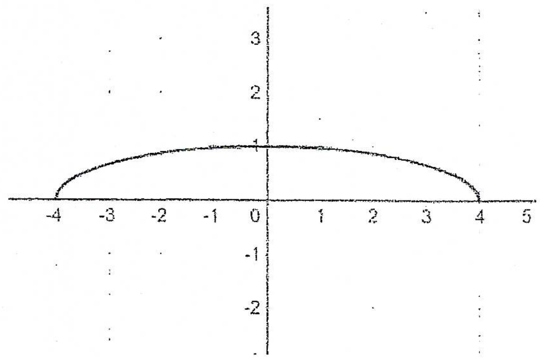
b) $y =$ ___________



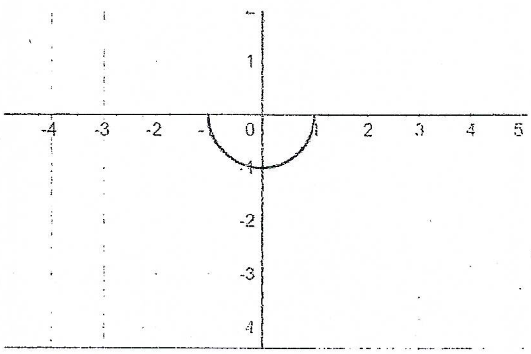
c) $y =$ ____________



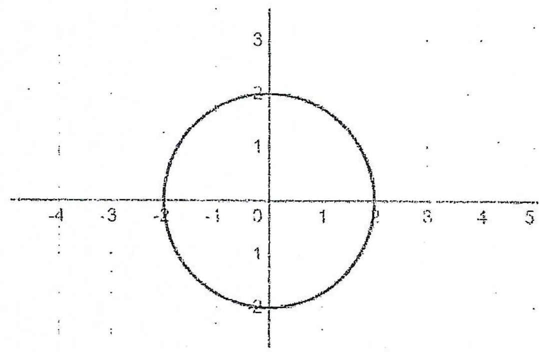
d) $y =$ ____________



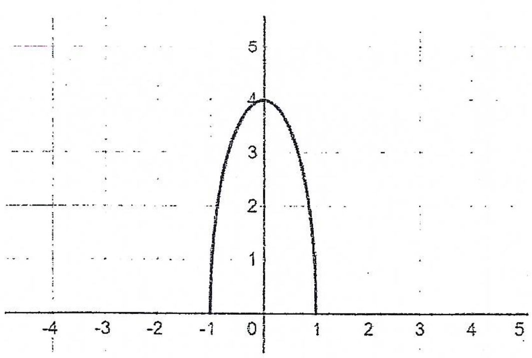
e) $y =$ ____________



f) $y =$ ____________



g) $y =$ ____________



h) $y =$ ____________